

# SIMPSON TYPE INEQUALITIES FOR FIRST ORDER DIFFERENTIABLE PREINVEX AND PREQUASIINVEX FUNCTIONS

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ABSTRACT. In this paper, we obtain some inequalities for functions whose first derivatives in absolute value are preinvex and prequasiinvex.

## 1. INTRODUCTION AND PRELIMINARIES

Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is a four times continuously differentiable mapping on  $(a, b)$  and  $\|f^{(4)}\|_{\infty} = \sup |f^{(4)}(x)| < \infty$ . The following inequality

$$\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4$$

is well known in the literature as Simpson's inequality.

For some results about Simpson inequality see [1]-[5].

Let  $K$  be a nonempty closed set in  $\mathbb{R}^n$ . We denote by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  the inner product and norm respectively. Let  $f : K \rightarrow \mathbb{R}$  and  $\eta : K \times K \rightarrow \mathbb{R}$  be continuous functions.

**Definition 1.** (See [7]) Let  $u \in K$ . Then the set  $K$  is said to be invex at  $u$  with respect to  $\eta(\cdot, \cdot)$ , if

$$u + t\eta(v, u) \in K, \quad \forall u, v \in K, \quad t \in [0, 1].$$

$K$  is said to be invex set with respect to  $\eta$ , if  $K$  is invex at each  $u \in K$ . The invex set  $K$  is also called a  $\eta$ -connected set.

**Remark 1.** (See [6]) We would like to mention that the Definition 1 of an invex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point  $u$  which is contained in  $K$ . We don't require that the point  $v$  should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that  $v$  should be an end point of the path for every pair of points,  $u, v \in K$ , then  $\eta(v, u) = v - u$  and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to  $\eta(v, u) = v - u$ , but the converse is not necessarily true.

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1991 *Mathematics Subject Classification.* 26D10, 26D15.

*Key words and phrases.* Simpson inequality, preinvex function, hölder inequality, power-mean inequality.

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**Definition 2.** (See [7]) The function  $f$  on the invex set  $K$  is said to be preinvex with respect to  $\eta$ , if

$$f(u + t\eta(v, u)) \leq (1-t)f(u) + tf(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

The function  $f$  is said to be preconcave if and only if  $-f$  is preinvex. Note that every convex function is a preinvex function, but the converse is not true. For example, the function  $f(u) = -|u|$  is not a convex function, but it is a preinvex function with respect to  $\eta$ , where

$$\eta(v, u) = \begin{cases} v - u, & \text{if } v \leq 0, u \leq 0 \quad \text{and} \quad v \geq 0, u \geq 0 \\ u - v, & \text{otherwise.} \end{cases}$$

**Definition 3.** (See [8]) The function  $f$  on the invex set  $K$  is said to be prequasiinvex with respect to  $\eta$ , if

$$f(u + t\eta(v, u)) \leq \max \{f(u), f(v)\}, \quad \forall u, v \in K, \quad t \in [0, 1].$$

In this paper, we establish some new Simpson type inequalities for preinvex and prequasiinvex functions.

## 2. SIMPSON TYPE INEQUALITIES FOR PREINVEX FUNCTIONS

We used the following Lemma to obtain our main results.

**Lemma 1.** Let  $I \subseteq \mathbb{R}$  be an open invex subset with respect to  $\eta : I \times I \rightarrow \mathbb{R}_+$  and  $f : I \rightarrow \mathbb{R}$  be an absolutely continuous mapping on  $I, a, b \in I$  with  $\eta(b, a) \neq 0$ . If  $f'$  is integrable on  $\eta$ -path  $P_{ac}$ ,  $c = a + \eta(b, a)$ , following equality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right| \\ &= \eta(b, a) \int_0^1 m(t) f'(a + t\eta(b, a)) dt, \end{aligned}$$

where

$$m(t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{2}) \\ t - \frac{5}{6}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

*Proof.* Since  $a, b \in I$  and  $I$  is an invex set with respect to  $\eta$ , it is obvious that  $a + t\eta(b, a) \in I$  for  $t \in [0, 1]$  Integrating by parts implies that

$$\begin{aligned} & \int_0^{\frac{1}{2}} \left(t - \frac{1}{6}\right) f'(a + t\eta(b, a)) dt + \int_{\frac{1}{2}}^1 \left(t - \frac{5}{6}\right) f'(a + t\eta(b, a)) dt \\ &= \left(t - \frac{1}{6}\right) \frac{f(a + t\eta(b, a))}{\eta(b, a)} \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{f(a + t\eta(b, a))}{\eta(b, a)} dt \\ & \quad + \left(t - \frac{5}{6}\right) \frac{f(a + t\eta(b, a))}{\eta(b, a)} \Big|_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 \frac{f(a + t\eta(b, a))}{\eta(b, a)} dt \\ &= \frac{1}{6\eta(b, a)} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] \\ & \quad - \frac{1}{\eta(b, a)} \left[ \int_0^{\frac{1}{2}} f(a + t\eta(b, a)) dt + \int_{\frac{1}{2}}^1 f(a + t\eta(b, a)) dt \right]. \end{aligned}$$

If we change the variable  $x = a + t\eta(b, a)$  and multiply the resulting equality with  $\eta(b, a)$  we get the desired result.  $\square$

**Theorem 1.** *Let  $I \subseteq \mathbb{R}$  be an open invex subset with respect to  $\eta : I \times I \rightarrow \mathbb{R}_+$  and  $f : I \rightarrow \mathbb{R}$  be an absolutely continuous mapping on  $I, a, b \in I$  with  $\eta(b, a) \neq 0$ . If  $|f'|$  is preinvex then the following inequality holds:*

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right| \\ & \leq \frac{5}{72} \eta(b, a) [|f'(a)| + |f'(b)|]. \end{aligned}$$

*Proof.* From Lemma 1 and using the preinvexity of  $|f'|$  we have

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right| \\ & \leq \eta(b, a) \left\{ \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'(a + t\eta(b, a))| dt + \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| |f'(a + t\eta(b, a))| dt \right\} \\ & \leq \eta(b, a) \left\{ \int_0^{\frac{1}{6}} \left( \frac{1}{6} - t \right) [(1-t)|f'(a)| + t|f'(b)|] dt \right. \\ & \quad + \int_{\frac{1}{6}}^{\frac{1}{2}} \left( t - \frac{1}{6} \right) [(1-t)|f'(a)| + t|f'(b)|] dt \\ & \quad + \int_{\frac{1}{2}}^{\frac{5}{6}} \left( \frac{5}{6} - t \right) [(1-t)|f'(a)| + t|f'(b)|] dt \\ & \quad \left. + \int_{\frac{5}{6}}^1 \left( t - \frac{5}{6} \right) [(1-t)|f'(a)| + t|f'(b)|] dt \right\}. \end{aligned}$$

If we compute the above integrals, we get the desired result.  $\square$

**Theorem 2.** *Let  $I \subseteq \mathbb{R}$  be an open invex subset with respect to  $\eta : I \times I \rightarrow \mathbb{R}_+$  and  $f : I \rightarrow \mathbb{R}$  be an absolutely continuous mapping on  $I, a, b \in I$  with  $\eta(b, a) \neq 0$ . If  $|f'|$  is preinvex for some fixed  $q > 1$  then the following inequality holds*

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right| \\ & \leq \eta(b, a) \left( \frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left( \frac{3}{8} |f'(a)|^q + \frac{1}{8} |f'(b)|^q \right)^{\frac{1}{q}} + \left( \frac{1}{8} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where  $p = \frac{q}{q-1}$ .

*Proof.* From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \eta(b, a) \left\{ \left( \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{2}} |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{2}}^1 |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Since  $|f'|^q$  is preinvex, we obtain

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \eta(b, a) \left\{ \left( \int_0^{\frac{1}{6}} \left( \frac{1}{6} - t \right)^p dt + \int_{\frac{1}{6}}^{\frac{1}{2}} \left( t - \frac{1}{6} \right)^p dt \right)^{\frac{1}{p}} \right. \\ & \quad \times \left( \int_0^{\frac{1}{2}} [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \right)^{\frac{1}{q}} \\ & \quad + \left( \int_{\frac{1}{2}}^{\frac{5}{6}} \left( \frac{5}{6} - t \right)^p dt + \int_{\frac{5}{6}}^1 \left( t - \frac{5}{6} \right)^p dt \right)^{\frac{1}{p}} \\ & \quad \times \left( \int_{\frac{1}{2}}^1 [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \right)^{\frac{1}{q}} \Big\} \\ & = \eta(b, a) \left( \frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left( \frac{3}{8} |f'(a)|^q + \frac{1}{8} |f'(b)|^q \right)^{\frac{1}{q}} + \left( \frac{1}{8} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

The proof is completed.  $\square$

**Theorem 3.** *Under the assumptions of Theorem 2, we have the following inequality*

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \eta(b, a) \left( \frac{2(1 + 2^{p+1})}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[ \frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}. \end{aligned}$$

*Proof.* From Lemma 1, preinvexity of  $|f'|^q$  and using the Hölder inequality, we have

$$\begin{aligned}
& \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\
& \leq \eta(b, a) \left[ \int_0^1 |m(t)| |f'(a + t\eta(b, a))| dt \right] \\
& \leq \eta(b, a) \left( \int_0^1 |m(t)|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\
& \leq \eta(b, a) \left( \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt + \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \right)^{\frac{1}{q}} \\
& = \eta(b, a) \left( \frac{2(1+2^{p+1})}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[ \frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}
\end{aligned}$$

where we used the fact that

$$\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt = \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt = \frac{(1+2^{p+1})}{6^{p+1}(p+1)}.$$

The proof is completed.  $\square$

**Theorem 4.** Let  $I \subseteq \mathbb{R}$  be an open invex subset with respect to  $\eta : I \times I \rightarrow \mathbb{R}_+$  and  $f : I \rightarrow \mathbb{R}$  be an absolutely continuous mapping on  $I, a, b \in I$  with  $\eta(b, a) \neq 0$ . If  $|f'|$  is preinvex for some fixed  $q \geq 1$  then the following inequality holds

$$\begin{aligned}
& \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\
& \leq \eta(b, a) \left( \frac{5}{72} \right)^{1-\frac{1}{q}} \\
& \quad \times \left\{ \left( \frac{61|f'(a)|^q + 29|f'(b)|^q}{1296} \right)^{\frac{1}{q}} + \left( \frac{29|f'(a)|^q + 61|f'(b)|^q}{1296} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

*Proof.* From Lemma 1 and using the power-mean inequality, we have

$$\begin{aligned}
& \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\
& \leq \eta(b, a) \\
& \quad \times \left\{ \left( \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| dt \right)^{1-\frac{1}{q}} \left( \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left( \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| dt \right)^{1-\frac{1}{q}} \left( \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Since  $|f'|^q$  is preinvex function we have

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'(a + t\eta(b, a))|^q dt \\
& \leq \int_0^{\frac{1}{6}} \left( \frac{1}{6} - t \right) [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \\
& \quad + \int_{\frac{1}{6}}^{\frac{1}{2}} \left( t - \frac{1}{6} \right) [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \\
& = \frac{61|f'(a)|^q + 29|f'(b)|^q}{1296}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| |f'(a + t\eta(b, a))|^q dt \\
& \leq \int_{\frac{1}{2}}^{\frac{5}{6}} \left( \frac{5}{6} - t \right) [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \\
& \quad + \int_{\frac{5}{6}}^1 \left( t - \frac{5}{6} \right) [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \\
& = \frac{29|f'(a)|^q + 61|f'(b)|^q}{1296}.
\end{aligned}$$

Combining all the above inequalities gives us the desired result.  $\square$

### 3. SIMPSON TYPE INEQUALITIES FOR PREQUASINVEX FUNCTIONS

In this section, we obtained Simpson type inequalities for prequasiinvex functions.

**Theorem 5.** *Let  $I \subseteq \mathbb{R}$  be an open invex subset with respect to  $\eta : I \times I \rightarrow \mathbb{R}_+$  and  $f : I \rightarrow \mathbb{R}$  be an absolutely continuous mapping on  $I, a, b \in I$  with  $\eta(b, a) \neq 0$ . If  $|f'|$  is prequasiinvex for some fixed  $q \geq 1$  then the following inequality holds*

$$\begin{aligned}
& \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\
& \leq \frac{5}{36} \eta(b, a) [\max\{|f'(a)|^q, |f'(b)|^q\}]^{\frac{1}{q}}.
\end{aligned}$$

*Proof.* From Lemma 1 and using the power-mean inequality, we have

$$\begin{aligned}
& \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\
& \leq \eta(b, a) \\
& \quad \times \left\{ \left( \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| dt \right)^{1-\frac{1}{q}} \left( \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left( \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| dt \right)^{1-\frac{1}{q}} \left( \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Since  $|f'|^q$  is prequasiinvex function we have

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'(a + t\eta(b, a))|^q dt \\
& \leq \int_0^{\frac{1}{6}} \left( \frac{1}{6} - t \right) [\max \{|f'(a)|^q, |f'(b)|^q\}] dt \\
& \quad + \int_{\frac{1}{6}}^{\frac{1}{2}} \left( t - \frac{1}{6} \right) [\max \{|f'(a)|^q, |f'(b)|^q\}] dt \\
& = \frac{5}{72} [\max \{|f'(a)|^q, |f'(b)|^q\}]
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| |f'(a + t\eta(b, a))|^q dt \\
& \leq \int_{\frac{1}{2}}^{\frac{5}{6}} \left( \frac{5}{6} - t \right) [\max \{|f'(a)|^q, |f'(b)|^q\}] dt \\
& \quad + \int_{\frac{5}{6}}^1 \left( t - \frac{5}{6} \right) [\max \{|f'(a)|^q, |f'(b)|^q\}] dt \\
& = \frac{5}{72} [\max \{|f'(a)|^q, |f'(b)|^q\}].
\end{aligned}$$

From the above inequalities we get the desired result.  $\square$

**Corollary 1.** In Theorem 5, if we choose  $q = 1$  we obtain

$$\begin{aligned}
& \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right| \\
& \leq \frac{5}{36} \eta(b, a) \max \{|f'(a)|, |f'(b)|\}.
\end{aligned}$$

**Corollary 2.** In Theorem 5, if we choose  $f(a) = f\left(\frac{2a + \eta(b, a)}{2}\right) = f(a + \eta(b, a))$  we obtain the midpoint type inequality as follows:

$$\begin{aligned}
& \left| f\left(\frac{2a + \eta(b, a)}{2}\right) - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right| \\
& \leq \frac{5}{36} \eta(b, a) \max \{|f'(a)|, |f'(b)|\}.
\end{aligned}$$

**Theorem 6.** Let  $I \subseteq \mathbb{R}$  be an open invex subset with respect to  $\eta : I \times I \rightarrow \mathbb{R}_+$  and  $f : I \rightarrow \mathbb{R}$  be an absolutely continuous mapping on  $I, a, b \in I$  with  $\eta(b, a) \neq 0$ .

If  $|f'|$  is prequasiinvex for some fixed  $q > 1$  then the following inequality holds

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq 2\eta(b, a) \left( \frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left( \frac{\max\{|f'(a)|^q, |f'(b)|^q\}}{2} \right)^{\frac{1}{q}} \end{aligned}$$

where  $p = \frac{q}{q-1}$ .

*Proof.* From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \eta(b, a) \left\{ \left( \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{2}} |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{2}}^1 |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Since  $|f'|^q$  is prequasiinvex, we obtain

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \eta(b, a) \left\{ \left( \int_0^{\frac{1}{6}} \left( \frac{1}{6} - t \right)^p dt + \int_{\frac{1}{6}}^{\frac{1}{2}} \left( t - \frac{1}{6} \right)^p dt \right)^{\frac{1}{p}} \right. \\ & \quad \times \left( \int_0^{\frac{1}{2}} \max\{|f'(a)|^q, |f'(b)|^q\} dt \right)^{\frac{1}{q}} \\ & \quad + \left( \int_{\frac{1}{2}}^{\frac{5}{6}} \left( \frac{5}{6} - t \right)^p dt + \int_{\frac{5}{6}}^1 \left( t - \frac{5}{6} \right)^p dt \right)^{\frac{1}{p}} \\ & \quad \times \left( \int_{\frac{1}{2}}^1 \max\{|f'(a)|^q, |f'(b)|^q\} dt \right)^{\frac{1}{q}} \Big\} \\ & = 2\eta(b, a) \left( \frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left( \frac{\max\{|f'(a)|^q, |f'(b)|^q\}}{2} \right)^{\frac{1}{q}}. \end{aligned}$$

We used

$$\begin{aligned} \int_0^{\frac{1}{6}} \left( \frac{1}{6} - t \right)^p dt + \int_{\frac{1}{6}}^{\frac{1}{2}} \left( t - \frac{1}{6} \right)^p dt &= \int_{\frac{1}{2}}^{\frac{5}{6}} \left( \frac{5}{6} - t \right)^p dt + \int_{\frac{5}{6}}^1 \left( t - \frac{5}{6} \right)^p dt \\ &= \frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \end{aligned}$$

in the above inequality to complete the proof.  $\square$



**Theorem 7.** *Under the assumptions of Theorem 6, we have the following inequality*

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \eta(b, a) \left( \frac{2(1 + 2^{p+1})}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[ \frac{\max \{|f'(a)|^q, |f'(b)|^q\}}{2} \right]^{\frac{1}{q}}. \end{aligned}$$

*Proof.* From Lemma 1, prequasiinvexity of  $|f'|^q$  and using the Hölder inequality, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \eta(b, a) \left[ \int_0^1 |m(t)| |f'(a + t\eta(b, a))| dt \right] \\ & \leq \eta(b, a) \left( \int_0^1 |m(t)|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \max \{|f'(a)|^q, |f'(b)|^q\} dt \right)^{\frac{1}{q}} \\ & \leq \eta(b, a) \left( \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt + \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \max \{|f'(a)|^q, |f'(b)|^q\} dt \right)^{\frac{1}{q}} \\ & = \eta(b, a) \left( \frac{2(1 + 2^{p+1})}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[ \frac{\max \{|f'(a)|^q, |f'(b)|^q\}}{2} \right]^{\frac{1}{q}} \end{aligned}$$

where we used the fact that

$$\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt = \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt = \frac{(1 + 2^{p+1})}{6^{p+1}(p+1)}.$$

The proof is completed.  $\square$

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